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On Surface Interpolation

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INTRODUCTION

It is the aim of this paper to study the method of surface interpolation introduced by Coons [3] and Gordon [5] in the abstract setting of optimal interpolation in the sense of Sard [7, 8]. As an illustration, blended linear interpolation will be considered (Gordon [4], Birkhoff and Gordon [2]).

1. OPTIMAL INTERPOLATION

The theory of optimal interpolation as a special case of Sard's theory of optimal approximation [7, 8] is based on the tuple

$$(X, Y, Z, U, F). \tag{1.1}$$

Here X, Y, Z are complex (separable) Hilbert spaces and

$$U: X \to Y, \qquad F: X \to Z$$

denote continuous linear mappings. Let us suppose that the completeness condition holds (Sard [8]). Thus, the bilinear form

$$((x, y)) = (Ux, Uy) + (Fx, Fy) \qquad (x, y \in X) \tag{1.2}$$

is a scalar product on X, which induces the original topology on X.

The problem of *optimal interpolation* reads as follows.

For every $x \in X$ determine $\xi \in X$ such that

$$F\xi = Fx, \qquad \| U\xi \| \leqslant \| Uy \| \qquad (Fy = Fx), \qquad (y \in X). \tag{1.3}$$

Following Sard [7, 8], we construct the orthogonal projector P on $(X; ((.,.)))$ defined by

$$\mathrm{Im}(P) = \mathrm{Ker}(F)^{\perp}. \tag{1.4}$$

Then the element

$$\xi = Px$$

is the unique solution of (1.3). The operator P is called the *spline interpolation projector* corresponding to (1.1).

2. Abstract Surface Interpolation

The basic implicit tool of the method of surface interpolation of Coons and Gordon is the tensor product method. For tensor products of Hilbert spaces and linear mappings we refer to Berezanskij [1, pp. 39–50].

Let us consider two problems of optimal interpolation that are described by

$$(X_j, Y_j, Z_j, U_j, F_j) \qquad (j = 1, 2).$$

Denote by

$$P_j \qquad (j = 1, 2)$$

the corresponding spline interpolation projectors and let

$$I_j \qquad (j = 1, 2)$$

denote the identity mapping of X_j ($j = 1, 2$). Now we are able to state the problem of *abstract surface interpolation:*

For every $x \in X_1 \otimes X_2$ determine $\xi \in X_1 \otimes X_2$ such that

$$F_1 \otimes I_2(\xi) = F_1 \otimes I_2(x), \qquad I_1 \otimes F_2(\xi) = I_1 \otimes F_2(x),$$

$$\| U_1 \otimes U_2(\xi)\| \leqslant \| U_1 \otimes U_2(y)\|,$$

$$F_1 \otimes I_2(y) = F_1 \otimes I_2(x), \qquad I_1 \otimes F_2(y) = I_1 \otimes F_2(x) \qquad (y \in X_1 \otimes X_2).$$

It is clear that the problem of abstract surface interpolation can be considered as a problem of optimal interpolation which is characterized by the tuple

$$(X_1 \otimes X_2, Y_1 \otimes Y_2, Z_1 \otimes X_2 \times X_1 \otimes Z_2, U_1 \otimes U_2, F_1 \otimes I_2 \times I_1 \otimes F_2). \tag{2.1}$$

More precisely, we have the following

THEOREM. *Suppose that*

$$\operatorname{Im}(F_j) = Z_j \qquad (j = 1, 2). \tag{2.2}$$

Then for every $x \in X_1 \otimes X_2$ the element

$$\xi = (P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2)(x) \tag{2.3}$$

is the unique solution of the problem of abstract surface interpolation.

Proof. First let us verify the completeness condition (1.2) for the tuple (2.1). This condition is a direct consequence of the scalar product of the tensor product of $(X_1; ((.,.)))$ and $(X_2; ((.,.)))$, since we have for any pair $x, y \in X_1 \otimes X_2$:

$$\begin{aligned}(x, y) &= (F_1 \otimes F_2(x), F_1 \otimes F_2(y)) + (F_1 \otimes U_2(x), F_1 \otimes U_2(y)) \\ &\quad + (U_1 \otimes F_2(x), U_1 \otimes F_2(y)) + (U_1 \otimes U_2(x), U_1 \otimes U_2(y)),\end{aligned}$$

$$\begin{aligned}(F_1 \otimes I_2(x), F_1 \otimes I_2(y)) &= (F_1 \otimes F_2(x), F_1 \otimes F_2(y)) \\ &\quad + (F_1 \otimes U_2(x), F_1 \otimes U_2(y)),\end{aligned}$$

$$\begin{aligned}(I_1 \otimes F_2(x), I_1 \otimes F_2(y)) &= (F_1 \otimes F_2(x), F_1 \otimes F_2(y)) \\ &\quad + (U_1 \otimes F_2(x), U_1 \otimes F_2(y)),\end{aligned}$$

$$((x, y)) = (x, y) + (F_1 \otimes F_2(x), F_1 \otimes F_2(y)).$$

Observe that the scalar product of the tensor product of $(X_1; ((.,.)))$ and $(X_2; ((.,.)))$ and the scalar product of the tensor product of $(X_1; (.,.))$ and $(X_2; (.,.))$ induce equivalent norms. Thus the completeness condition holds in both cases.

The more difficult part of the proof is to show that the operator

$$P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2$$

is the spline interpolation projector corresponding to (2.1). Taking into account that

$$F_j P_j = F_j \qquad (j = 1, 2),$$

$$P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2 = I_1 \otimes I_2 - (I_1 - P_1) \otimes (I_2 - P_2),$$

it follows for every $x \in X_1 \otimes X_2$ (for ξ see (2.3)):

$$(\xi, x - \xi) = ((\xi, x - \xi)) = (U_1 \otimes U_2(\xi), U_1 \otimes U_2(x - \xi)) = 0.$$

Thus $P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2$ is an orthogonal projector on $(X_1 \otimes X_2; ((.,.)))$ that satisfies

$$(F_1 \otimes I_2)(P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2) = F_1 \otimes I_2,$$

$$(I_1 \otimes F_2)(P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2) = I_1 \otimes F_2.$$

Hence

$$\operatorname{Ker}(P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2) \subset \operatorname{Ker}(F_1 \otimes I_2 \times I_1 \otimes F_2).$$

The only thing we have still to prove is to show that for every $x \in X_1 \otimes X_2$ the relations

$$F_1 \otimes I_2(x) = 0, \qquad I_1 \otimes F_2(x) = 0$$

imply

$$(P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2)(x) = 0.$$

If we take into account (1.4) and (2.2), an application of Banach's theorem yields the existence of the continuous linear mappings

$$F_j^{-1}: Z_j \to X_j \qquad (j = 1, 2)$$

that satisfy

$$F_j^{-1} F_j = P_j \qquad (j = 1, 2).$$

Thus we have

$$(F_1^{-1} \otimes I_2)(F_1 \otimes I_2) = P_1 \otimes I_2, \qquad (I_1 \otimes F_2^{-1})(I_1 \otimes P_2) = I_1 \otimes P_2.$$

Hence the relations

$$F_1 \otimes I_2(x) = 0, \qquad I_1 \otimes F_2(x) = 0$$

imply

$$P_1 \otimes I_2(x) = 0, \qquad I_1 \otimes P_2(x) = 0.$$

Since we have

$$(P_1 \otimes I_2)(I_1 \otimes P_2) = P_1 \otimes P_2,$$

we obtain

$$(P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2)(x) = 0.$$

Remark. Note that the projector $P_1 \otimes I_2 + I_1 \otimes P_2 - P_1 \otimes P_2$ is the Boolean sum of the projectors $P_1 \otimes I_2$ and $I_1 \otimes P_2$. For a further discussion of this aspect we refer to [6].

3. An Example

One way of describing *linear interpolation* as optimal interpolation is based on the choice of the tuple

$$(W^{1,2}(J), L^2(J), C^2, D, \epsilon_0 \times \epsilon_1) \tag{3.1}$$

($J = [0, 1]$, $W^{1,2}(J)$ a Sobolev space, ϵ_0, ϵ_1 Dirac measures). In this connection the linear interpolant of the function $f \in W^{1,2}(J)$,

$$\xi(s) = f(0)(1 - s) + f(1)s,$$

is the unique function in the set of functions $g \in W^{1,2}(J)$ satisfying

$$g(0) = f(0), \qquad g(1) = f(1),$$

which minimizes

$$\int_J |Dg(s)|^2 \, ds.$$

The *blended linear interpolation* [2, 4] is obtained by choosing (3.1) twice. An application of the general procedure of Section 2 yields the following concrete result (see also [4, 9]).

The blended linear interpolant of $f \in W^{1,2}(J) \otimes W^{1,2}(J)$,

$$\begin{aligned}\xi(s, t) = f(0, t)(1-s) + f(1, t)s + f(s, 0)(1-t) + f(s, 1)t \\ - f(0, 0)(1-s)(1-t) - f(0, 1)(1-s)t \\ - f(1, 0)\, s(1-t) - f(1, 1)\, st,\end{aligned}$$

is the unique function among all $g \in W^{1,2}(J) \otimes W^{1,2}(J)$ satisfying

$$g(.,0) = f(.,0), \qquad g(.,1) = f(.,1),$$

$$g(0,.) = f(0,.), \qquad g(1,.) = f(1,.),$$

which minimizes

$$\int_J \int_J | D_x D_y g(s, t)|^2 \, ds \, dt.$$

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